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On the Generating Functions for Certain Classes of Plane Partitions

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A plane partition π is said to be *H-invariant* if its diagram $D(\pi)$ is stable under H, where H is a subgroup of S_3 acting on $\mathbb{N}^3$ by permutating coordinates. In this paper we give explicit generating functions for A_3-invariant plane partitions and S_3-invariant plane partitions. Also we find formulas involving Pfaffians for the sum of all minors of an arbitrary matrix. We can use these formulas to obtain simple determinantal expressions for the generating function of S_3-invariant plane partitions and reduce the Andrews–Robbins conjecture to the evaluation of some matrix given in Theorem 5. © 1989 Academic Press, Inc.

INTRODUCTION

The purpose of this paper is to give explicit generating functions for plane partitions which have certain symmetries. In order to explain in detail, we shall prepare some notations. (cf. [St 1])

A *plane partition* is an array of positive integers

$$\begin{array}{llllll} a_{11} & a_{12} & a_{13} & \cdots & & a_{1\lambda_1} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2\lambda_2} & \\ & \cdots & & & & \\ a_{r1} & a_{r2} & \cdots & a_{r\lambda_r} & & \end{array}$$

which satisfies the following three conditions:

(P1) $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$

(P2) $a_{i,j} \geqslant a_{i,j+1}$ for $i=1, ..., r$ and $j=1, 2, ..., \lambda_i - 1$

(P3) $a_{i,j} \geqslant a_{i+1,j}$ for $i=1, ..., r-1$ and $j=1, 2, ..., \lambda_{i+1}$.

(The empty array $\varnothing$, which has no rows, is permitted.) With such an array $\pi = (a_{ij})$, we associate the *diagram* $D(\pi)$ defined by

$$D(\pi) = \{(i, j, k) \in \mathbb{N}^3;\ a_{ij} \text{ is defined and } 1 \leqslant k \leqslant a_{ij}\}.$$

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Then conditions (P1), (P2), and (P3) can be expressed as

(P) For (i, j, k), $(i', j', k') \in \mathbb{N}^3$ such that $i \geqslant i'$, $j \geqslant j'$ and $k \geqslant k'$; $(i, j, k) \in D(\pi)$ implies $(i', j', k') \in D(\pi)$.

Note that π is recovered uniquely from $D(\pi)$.

The symmetric group S_3 acts on $\mathbb{N}^3$ by permutating coordinates. For a plane partition π and $g \in S_3$, it follows from (P) that $g \cdot D(\pi)$ is the diagram of some plane partition. Let H be a subgroup of S_3. A plane partition π is called *H-invariant* if $D(\pi)$ is H-stable. We are interested in the cardinality of

$$\mathscr{P}(H; n) = \{\pi; \pi \text{ is an } H\text{-invariant plane partition such that } D(\pi) \text{ is contained in } [n]^3\},$$

where $[n] = \{1, 2, ..., n\}$.

DEFINITION. For a plane partition π, we set

$$|\pi| = \# D(\pi),$$

where $\# A$ denotes the cardinality of a set A. If H is a subgroup of S_3 and π is H-invariant, then we denote by $wt_H(\pi)$ the number of the H-orbits in $D(\pi)$. Moreover, we define

$$P(H; n) = \sum q^{|\pi|},$$

$$Q(H; n) = \sum q^{wt_H(\pi)},$$

where the sums are taken over all $\pi \in \mathscr{P}(H; n)$.

The main object of this paper is to give formulas for $Q(A_3; n)$, $P(S_3; n)$, and $Q(S_3; n)$. The formulas for other subgroups H are known and can be unified in the following form (see [St2]).

THEOREM. *Let H be a subgroup of S_3. For an H-orbit ξ through $x = (i, j, k) \in \mathbb{N}^3$, we put*

$$ht(\xi) = i + j + k - 2.$$

(a) *If H is not S_3, then*

$$P(H; n) = \prod_{\xi \in [n]^3/H} \frac{1 - q^{\# \xi \cdot (1 + ht(\xi))}}{1 - q^{\# \xi \cdot ht(\xi)}}. \tag{1}$$

(b) *If H is neither S_3 nor A_3, then*

$$Q(H; n) = \prod_{\xi \in [n]^3/H} \frac{1 - q^{1 + ht(\xi)}}{1 - q^{ht(\xi)}}. \tag{2}$$

Furthermore, G. E. Andrews [An3] and D. P. Robbins conjectured that (2) is valid for the case $H = S_3$. In the case $H = S_3$, (1) is not true because its right-hand side is not a polynomial. By the same reason, (2) is not valid for the case $H = A_3$. Since $P(S_3; n)$ and $Q(A_3; n)$ have irreducible factors which are not cyclotomic polynomials, they cannot be written in the form $\prod (1-q^{a_i})/\prod (1-q^{b_j})(a_i, b_j \in \mathbb{N})$.

In Section 1, we give a formula for $Q(A_3; n)$. For $P(S_3; n)$ or $Q(S_3; n)$, we shall show that it is equal to the sum of all minors (including the void minor equal to 1) of a certain lower triangular matrix. On the other hand, we find formulas involving a Pfaffian for the sum of all minors of an arbitrary matrix (see Theorem 4). We can use these formulas to obtain simple determinantal expressions for $P(S_3; n)^2$ and $Q(S_3; n)^2$. And the expression for $Q(S_3; n)^2$ reduces the above Andrews–Robbins conjecture to the evaluation of some matrix given in Theorem 5 of this paper.

The author is grateful to D. P. Robbins for suggesting the simplification of the proofs of Theorem 3 and 4.

1. Generating Function for A_3-invariant Plane Partitions

In order to derive the formulas for $Q(A_3; n)$, $P(S_3; n)$, and $Q(S_3; n)$, we make use of shifted plane partition defined as follows. A *shifted plane partition* is an array of positive integers

$$\begin{array}{ccccccccccc} b_{11} & b_{12} & b_{13} & \cdots & & & & & & & b_{1\mu_1} \\ & b_{22} & b_{23} & \cdots & & & & & & b_{2\mu_2} & \\ & & b_{33} & \cdots & & & & & b_{3\mu_3} & & \\ & & & \cdots & & & & & & & \\ & & & & b_{rr} & \cdots & b_{r\mu_r} & & & & \end{array}$$

satisfying

(S1) $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots \geqslant \mu_r$

(S2) $b_{i,j} \geqslant b_{i,j+1}$ for $i = 1, ..., r$ and $j = i, i+1, ..., \mu_i - 1$

(S3) $b_{i,j} \geqslant b_{i+1,j}$ for $i = 1, ..., r-1$ and $j = i, i+1, ..., \mu_{i+1}$.

(The empty array $\varnothing$ is permitted.) A shifted plane partition $\sigma = (b_{ij})$ is said to be *row-strict* (resp. *column-strict*) if σ satisfies

(S2)′ $b_{ij} > b_{i,j+1}$ (whenever both sides are defined) (resp.

(S3)′ $b_{ij} > b_{i+1,j}$ (whenever both side are defined)).

As in the case of plane partition, we define the shifted diagram $\mathrm{SD}(\sigma)$ of σ by setting

$$\mathrm{SD}(\sigma)=\{(i,j,k)\in\mathbb{N}^3;\, b_{ij} \text{ is defined (hence } i\leqslant j) \text{ and } 1\leqslant k\leqslant b_{ij}\}.$$

If σ has r rows and there are λ_i entries in the ith row, we denote by $\mathrm{sh}(\sigma)$ the partition $(\lambda_1, \lambda_2, ..., \lambda_r)$, and call it the *shape* of σ. Condition (S1) implies that $\lambda_1>\lambda_2>\cdots>\lambda_r$. And we define the *profile* $\mathrm{pr}(\sigma)$ of σ to be the partition $(b_{11}, b_{22}, ..., b_{rr})$. If σ is row-strict or column-strict, then $b_{11}>b_{22}>\cdots>b_{rr}$. For the empty shifted plane partition $\varnothing$, we put $\mathrm{sh}(\varnothing)=\mathrm{pr}(\varnothing)=(0)$ (the unique partition of 0). For example, if

$$\sigma=\begin{matrix} 6 & 6 & 3 & 2 & 1 \\ & 5 & 2 & 2 & \\ & & 2 & 1 & \\ & & & 1 & \end{matrix}$$

then $\mathrm{sh}(\sigma)=(5, 3, 2, 1)$, $\mathrm{pr}(\sigma)=(6, 5, 2, 1)$.

Now we shall consider $Q(A_3; n)$. Let $\mathscr{C}(n)$ be the set of all column-strict shifted plane partitions $\sigma=(b_{ij})$ such that $\mathrm{sh}(\sigma)=\mathrm{pr}(\sigma)$ and $b_{ij}\leqslant n$.

PROPOSITION 1. *For $\pi=(a_{ij})\in\mathscr{P}(A_3; n)$, we set*

$$b_{ij}=a_{ij}-i+1$$

only when $i\leqslant j$ and $a_{ij}\geqslant i$, and denote by $\gamma(\pi)$ the arrangement of the b_{ij}'s in the form

$$\gamma(\pi)=\begin{matrix} b_{11} & b_{12} & \cdots & \\ & b_{22} & \cdots & \\ & & \cdots & \\ & & b_{rr} & \cdots \end{matrix}$$

Then

(a) $\gamma(\pi)\in\mathscr{C}(n)$.

(b) *The map* $\gamma: \mathscr{P}(A_3; n)\to\mathscr{C}(n)$ *is a bijection.*

(c) $|\pi|=\sum_{i,j} b_{ij}$.

(d) $wt_{A_3}(\pi)=r+\sum_{i<j} b_{ij}$,

where r is the numer of rows in $\gamma(\pi)$.

Proof. (a), (b), and (c) have been proved in [MRR].

(d) If we set, for $m \in \mathbb{N}$,

$$G_m = \{(i, j, k) \in \mathbb{N}^3; \min(i, j, k) = m\},$$
$$H_m = \{(i, j, k) \in G_m; i = m\},$$

then b_{ij} is given by

$$b_{ij} = \min\{k; (i, j, k) \in D(\pi) \cap H_i\} - i + 1.$$

(If $\{k; (i, j, k) \in D(\pi) \cap H_i\} = \emptyset$, b_{ij} is not defined.) Since (i, j, k) $(j \geqslant i+1, i \leqslant k \leqslant b_{ij} + i - 1)$ and (i, i, i) are the representatives of A_3-orbits in $D(\pi) \cap G_i$, the numbers of A_3-orbits in $D(\pi) \cap G_i$ is equal to $1 + \sum_{j>i} b_{ij}$. Therefore, summing up for $i = 1, 2, ..., r$, we have $wt_{A_3}(\pi) = r + \sum_{i<j} b_{ij}$.

In consideration of part (d) of Proposition 1, for a shifted plane partition $\sigma = (b_{ij})$ consisting of r rows, we put $u(\sigma) = r + \sum_{i<j} b_{ij}$. Given two descending sequences $b_1 > b_2 > \cdots > b_r$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ of positive integers, let $h(b_1, b_2, ..., b_r; \lambda_1, \lambda_2, ..., \lambda_r; d)$ be the number of column-strict shifted plane partitions σ such that $\mathrm{pr}(\sigma) = (b_1, b_2, ..., b_r)$, $\mathrm{sh}(\sigma) = (\lambda_1, \lambda_2, ..., \lambda_r)$, and $u(\sigma) = d$. And put

$$h(b_1, ..., b_r; \lambda_1, ..., \lambda_r) = \sum_{d=0}^{\infty} h(b_1, ..., b_r; \lambda_1, ..., \lambda_r; d)\, q^d.$$

PROPOSITION 2. *For $b_1 > b_2 > \cdots > b_r$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, we have*

$$h(b_1, ..., b_r; \lambda_1, ..., \lambda_r) = \det\left(q^{\lambda_j} \begin{bmatrix} b_i + \lambda_j - 2 \\ \lambda_j - 1 \end{bmatrix}\right)_{1 \leqslant i, j \leqslant r}, \tag{3}$$

where $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is a Gaussian polynomial defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} \dfrac{(1-q^a)(1-q^{a-1}) \cdots (1-q^{a-b+1})}{(1-q)(1-q^2) \cdots (1-q^b)}, & \textit{if} \quad 0 \leqslant b \leqslant a \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. When $r = 1$ and $\lambda_r = 1$, $\sigma = (b_1)$ is the only shifted plane partition σ such that $\mathrm{pr}(\sigma) = (b_1)$ and $\mathrm{sh}(\sigma) = (1)$. Hence we have

$$h(b_1, 1) = q = q^1 \begin{bmatrix} b_1 + 1 - 1 \\ 1 - 1 \end{bmatrix}.$$

We proceed by double induction on r and λ_r. Let σ be a column-strict shifted plane partition whose shape is $(\lambda_1, ..., \lambda_r)$ and whose profile is

$(b_1, ..., b_r)$. If we remove the first entries in each rows of σ, we obtain a column-strict shifted plane partition σ', whose shape is $(\lambda_1 - 1, ..., \lambda_r - 1)$ (if $\lambda_r > 1$) or $(\lambda_1 - 1, ..., \lambda_{r-1} - 1)$ (if $\lambda_r = 1$). Moreover, if we put $\mathrm{pr}(\sigma') = (c_1, ... c_r)$ (if $\lambda_r > 1$) or $(c_1, ..., c_{r-1})$ (if $\lambda_r = 1$), we see that $b_1 \geqslant c_1 > b_2 \geqslant c_2 > \cdots$ and that

$$u(\sigma) = \begin{cases} u(\sigma') + c_1 + \cdots + c_r & \text{if } \lambda_r > 1, \\ u(\sigma') + c_1 + \cdots + c_{r-1} + 1 & \text{if } \lambda_r = 1. \end{cases}$$

Therefore we have a recurrence

$$h(b_1, ..., b_r; \lambda_1, ..., \lambda_r)$$
$$= \begin{cases} \displaystyle\sum_{b_1 \geqslant c_1 > b_2 \geqslant c_2 > \cdots > b_r \geqslant c_r \geqslant 1} q^{c_1 + \cdots + c_r} \\ \qquad \times h(c_1, ..., c_r; \lambda_1 - 1, ..., \lambda_r - 1) \quad (\text{if } \lambda_r > 1) \\ \displaystyle\sum_{b_1 \geqslant c_1 > b_2 \geqslant c_2 > \cdots \geqslant c_{r-1} > b_r} q^{c_1 + \cdots + c_{r-1} + 1} \\ \qquad \times h(c_1, ..., c_{r-1}; \lambda_1 - 1, ..., \lambda_{r-1} - 1) \quad (\text{if } \lambda_r = 1) \end{cases}$$

It is enough to show that the right-hand side of (3) satisfies this recurrence. In the case $\lambda_r > 1$, we have

$$\sum q^{c_1 + \cdots + c_r} \det \left(q^{\lambda_j - 1} \begin{bmatrix} c_i + (\lambda_j - 1) - 2 \\ (\lambda_j - 1) - 1 \end{bmatrix} \right)_{1 \leqslant i, j \leqslant r}$$
$$= \det \left(\sum_{c_i = b_{i+1} + 1}^{b_i} q^{\lambda_j + c_i - 1} \begin{bmatrix} c_i + \lambda_j - 3 \\ \lambda_j - 2 \end{bmatrix} \right)_{1 \leqslant i, j \leqslant r},$$

where $b_{r+1} = 0$. Using a formula $\sum_{j=0}^{n} q^j \left[\begin{smallmatrix} m+j \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} m+n+1 \\ m+1 \end{smallmatrix}\right]$, we see that this determinant is equal to

$$\det \left(q^{\lambda_j} \begin{bmatrix} b_i + \lambda_j - 2 \\ \lambda_{j-1} \end{bmatrix} - q^{\lambda_j} \begin{bmatrix} b_{i+1} + \lambda_j - 2 \\ \lambda_j - 1 \end{bmatrix} \right)_{1 \leqslant i, j \leqslant r}$$
$$= \det \left(q^{\lambda_j} \begin{bmatrix} b_i + \lambda_j - 2 \\ \lambda_j - 1 \end{bmatrix} \right)_{1 \leqslant i, j \leqslant r}$$

by starting from the bottom row and adding each row to the one above it. In the case $\lambda_r = 1$, similarly we have

$$\sum_{b_1 \geqslant c_1 > b_2 \geqslant \cdots \geqslant c_{r-1} > b_r} q^{1 + c_1 + \cdots + c_{r-1}} \det \left(q^{\lambda_j - 1} \begin{bmatrix} c_i + \lambda_j - 3 \\ \lambda_j - 2 \end{bmatrix} \right)_{1 \leqslant i,j \leqslant r-1}$$
$$= \det \left(q^{\lambda_j} \begin{bmatrix} b_i + \lambda_j - 2 \\ \lambda_j - 1 \end{bmatrix} \right)_{1 \leqslant i, j \leqslant r},$$

if we note that

$$\begin{bmatrix} b_i+\lambda_r-2 \\ \lambda_r-1 \end{bmatrix}-\begin{bmatrix} b_{i+1}+\lambda_r-2 \\ \lambda_r-1 \end{bmatrix}=0 \ (i\leqslant r-1)$$

and

$$\begin{bmatrix} b_r+\lambda_r-2 \\ \lambda_r-1 \end{bmatrix}-\begin{bmatrix} b_{r+1}+\lambda_r-2 \\ \lambda_r-1 \end{bmatrix}=1.$$

From Proposition 1, it follows that

$$Q(A_3; n)=1+\sum h(b_1, ..., b_r; b_1, ..., b_r),$$

where the sum is taken over all decreasing sequences $(n\geqslant)$ $b_1>b_2>\cdots>b_r$ $(\geqslant 1)$. On the other hand, Proposition 2 says that $h(b_1, ..., b_r; b_1, ..., b_r)$ is the principal $r\times r$ minor obtained by picking up the b_ith rows and the b_jth columns of the $n\times n$ matrix $C_n=(q^j[{}^{i+j-2}_{j-1}])_{1\leqslant i,j\leqslant n}$. $Q(A_3; n)$ is equal to the sum of all principal minors (including the void minor equal to 1) of C_n. Therefore we obtain

THEOREM 1. *Let C_n be the $n\times n$ matrix $(q^j[{}^{i+j-2}_{j-1}])_{1\leqslant i,j\leqslant n}$. Then*

$$Q(A_3; n)=\det(I_n+C_n),$$

where I_n denotes the $n\times n$ identity matrix.

Remark. G. E. Andrews [An2] had shown that

$$P(A_3; n)=\det(I_n+C'_n),$$

where $C'_n=(q^{3j-2}[{}^{i+j-2}_{j-1}]_3)_{1\leqslant i,j\leqslant n}$ and $[{}^a_b]_3$ means the Gaussian polynomial with q replaced by q^3. Afterwards, W. H. Mills, D. P. Robbins, and H. Rumsey, Jr. [MRR] proved that det $(I_n+C'_n)$ equals the product

$$\prod_{\xi\in[n]^3/H}(1-q^{\#\xi\cdot(1+ht(\xi))})/(1-q^{\#\xi\cdot ht(\xi)}).$$

2. GENERATING FUNCTIONS FOR S_3-INVARIANT PLANE PARTITIONS

In this section, we consider $P(S_3; n)$ and $Q(S_3; n)$. For a positive integer n, let $\mathscr{R}(n)$ be the set of all row-strict shifted plane partitions $\sigma=(b_{ij})$ such that $b_{ij}\leqslant n$.

PROPOSITION 3. *For $\pi=(a_{ij})\in\mathscr{P}(S_3; n)$, define*

$$b_{ij}=a_{ij}-j+1$$

only when $i \leqslant j$ *and* $a_{ij} \geqslant j$, *and denote by* $\rho(\pi)$ *the arrangement of the* b_{ij}*'s in the form*

$$\rho(\pi)=\begin{array}{ccccc} b_{11} & b_{12} & b_{23} & \cdots & \\ & b_{22} & b_{23} & \cdots & \\ & & & \cdots & \\ & & & b_{rr} & \cdots \end{array}$$

Then

(a) $\rho(\pi) \in \mathscr{R}(n)$.

(b) *The map* $\rho: \mathscr{P}(S_3; n) \to \mathscr{R}(n)$ *is a bijection.*

(c) $|\pi| = -2r + 3\sum_{i,j} b_{ij} + \sum_{i<j} (b_{ij} - 1)$,

where r *is the number of rows in* $\rho(\pi)$.

(d) $wt_{S_3}(\pi) = \sum_{i,j} b_{ij}$.

Proof. (a) is clear.

(b) Let E (resp. F) be the set of all points $(i, j, k) \in \mathbb{N}^3$ such that $i \leqslant j$ (resp. $i \leqslant j \leqslant k$), and $\tilde{\rho}: F \to E$ be the map defined by

$$\tilde{\rho}(i, j, k) = (i, j, k - j + 1).$$

Then we see that

$$\tilde{\rho}(D(\pi) \cap F) = \mathrm{SD}(\rho(\pi)). \tag{4}$$

In order to show that ρ is bijective, we shall construct the inverse map η of ρ. For $\sigma \in \mathscr{R}(n)$, we put

$$D_0 = \{(i, j, k) \in F : \tilde{\rho}(i, j, k) \in \mathrm{SD}(\sigma)\},$$

and

$$D = \bigcup_{g \in S_3} g \cdot D_0.$$

By noting the row-strictness of σ, it can be checked that $(i, j, k) \in D$ implies $(i-1, j, k) \in D$ $(i > 1)$. Since D is S_3-stable, D satisfies the condition (P) mentioned in the Introduction. Hence there is a unique S_3-invariant plane partition $\eta(\sigma) \in \mathscr{P}(S_3; n)$ such that $D(\eta(\sigma))$ coincides with D. It follows from (4) that η is the inverse map of ρ.

(c) For $x = (i, j, k) \in F$, the relation between $\sharp S_3 \cdot x$ and $\tilde{\rho}(x)$ is

(i) if $i = j = k$, $\sharp S_3 \cdot x = 1$ and $\tilde{\rho}(x) = (i, i, 1)$

(ii) if $i = j < k$, $\sharp S_3 \cdot x = 3$ and $\tilde{\rho}(x) = (i, i, l)$ $(l > 1)$

(iii) if $i < j = k$, $\sharp S_3 \cdot x = 3$ and $\tilde{\rho}(x) = (i, j, 1)$ $(i < j)$

(iv) if $i < j < k$, $\sharp S_3 \cdot x = 6$ and $\tilde{\rho}(x) = (i, j, l)$ $(i < j, l > 1)$.

Since $D(\pi)=\bigcup_{x\in D(\pi)\cap F} S_3\cdot x$ (disjoint union), we have

$$\begin{aligned}|\pi| &= \sharp\{(i,i,1)\in \mathrm{SD}(\rho(\pi))\}\\ &\quad +3\cdot\sharp\{(i,i,l)\in \mathrm{SD}(\rho(\pi));\, l>1\}\\ &\quad +3\cdot\sharp\{(i,j,1)\in \mathrm{SD}(\rho(\pi));\, i<j\}\\ &\quad +6\cdot\sharp\{(i,j,l)\in \mathrm{SD}(\rho(\pi));\, i<j,\, l>1\}\\ &= r+3\sum_{i=1}^{r}(b_{ii}-1)+3\,\sharp\{(i,j);\, b_{ij} \text{ is defined}, i<j\}\\ &\quad +6\sum_{i<j}(b_{ij}-1)\\ &= -2r+3\sum_{i,j} b_{ij}+3\sum_{i<j}(b_{ij}-1).\end{aligned}$$

(d) Since $D(\pi)\cap F$ is the system of representatives of S_3-orbits in $D(\pi)$,

$$wt_{S_3}(\pi)=\sharp(D(\pi)\cap F)=\sharp\mathrm{SD}(\rho(\pi))=\sum_{i,j} b_{ij}.$$

In consideration of (c) and (d) of Proposition 3, we define as follows: For a shifted plane partition $\sigma=(b_{ij})$ with r rows, we put

$$v(\sigma)=-2r+3\sum_{i,j} b_{ij}+3\sum_{i<j}(b_{ij}-1),$$

$$w(\sigma)=\sum_{i,j} b_{ij}.$$

Given two decreasing sequences $b_1>b_2>\cdots>b_r$ and $\lambda_1>\lambda_2>\cdots>\lambda_r$, let $f(b_1,\ldots,b_r;\lambda_1,\ldots,\lambda_r;d)$ (resp. $g(b_1,\ldots,b_r;\lambda_1,\ldots,\lambda_r;d)$) be the number of the row-strict shifted plane partitions σ such that $\mathrm{pr}(\sigma)=(b_1,\ldots,b_r)$, $\mathrm{sh}(\sigma)=(\lambda_1,\ldots,\lambda_r)$, and $v(\sigma)=d$ (resp. $w(\sigma)=d$). And put

$$f(b_1,\ldots,b_r;\lambda_1,\ldots,\lambda_r)=\sum_{d=0}^{\infty} f(b_1,\ldots,b_r;\lambda_1,\ldots,\lambda_r;d)\,q^d$$

$$g(b_1,\ldots,b_r;\lambda_1,\ldots,\lambda_r)=\sum_{d=0}^{\infty} g(b_1,\ldots,b_r;\lambda_1,\ldots,\lambda_r;d)\,q^d.$$

Then we obtain the following determinantal formulas.

PROPOSITION 4. *For $b_1>\cdots>b_r$ and $\lambda_1>\cdots>\lambda_r$, we have*

(a)

$$f(b_1,\ldots,b_r;\lambda_1,\ldots,\lambda_r)=\det\left(q^{3b_i+3\lambda_j-6\lambda_j^2+1}\begin{bmatrix} b_i-1\\ \lambda_j-1\end{bmatrix}_6\right)_{1\leqslant i,j\leqslant r}$$

(b)

$$g(b_1, ..., b_r; \lambda_1, ..., \lambda_r) = \det\left(q^{b_i + \binom{\lambda_j}{2}} \begin{bmatrix} b_i - 1 \\ \lambda_j - 1 \end{bmatrix}\right)_{1 \leqslant i,j \leqslant r},$$

where $[{a \atop b}]_6$ *denotes the Gaussian polynomial replaced* q *by* q^6.

Proof. These equalities can be proved by the same method that was used in the proof of Proposition 2, if we note the following recurrences:

$$f(b_1, ..., b_r; \lambda_1, ..., \lambda_r)$$

$$= \begin{cases} \displaystyle\sum_{b_1 > c_1 \geqslant b_2 > c_2 \geqslant \cdots b_r > c_r \geqslant 1} q^{3\sum_{i=1}^r b_i + 3\sum_{i=1}^r (c_i - 1)} \\ \qquad \times f(c_1, ..., c_r; \lambda_1 - 1, ..., \lambda_r - 1) \quad (\text{if } \lambda_r > 1) \\ \displaystyle\sum_{b_1 > c_1 \geqslant b_2 > \cdots \geqslant b_{r-1} > c_{r-1} \geqslant b_r} q^{-2 + 3\sum_{i=1}^r b_i + 3\sum_{i=1}^{r-1} (c_i - 1)} \\ \qquad \times f(c_1, ..., c_{r-1}; \lambda_1 - 1, ..., \lambda_{r-1} - 1) \quad (\text{if } \lambda_r = 1) \end{cases}$$

$$g(b_1, ..., b_r; \lambda_1, ..., \lambda_r)$$

$$= \begin{cases} \displaystyle\sum_{b_1 > c_1 \geqslant b_2 > c_2 \geqslant \cdots \geqslant b_r > c_r \geqslant 1} q^{b_1 + \cdots + b_r} \\ \qquad \times g(c_1, ..., c_r; \lambda_1 - 1, ..., \lambda_r - 1) \quad (\text{if } \lambda_r > 1) \\ \displaystyle\sum_{b_1 > c_1 \geqslant b_2 > \cdots \geqslant b_{r-1} > c_{r-1} \geqslant b_r} q^{b_1 + \cdots + b_r} \\ \qquad \times g(c_1, ..., c_{r-1}; \lambda_1 - 1, ..., \lambda_{r-1} - 1) \quad (\text{if } \lambda_r = 1) \end{cases}$$

From Proposition 3, it follows that

$$P(S_3; n) = 1 + \sum f(b_1, ..., b_r; \lambda_1, ..., \lambda_r),$$

$$Q(S_3; n) = 1 + \sum f(b_1, ..., b_r; \lambda_1, ..., \lambda_r),$$

where the sums are taken over all decreasing sequences $(n \geqslant)\ b_1 > \cdots > b_r$ $(\geqslant 1)$ and $(n \geqslant)\ \lambda_1 > \cdots > \lambda_r$ $(\geqslant 1)$ of arbitrary length r. On the other hand, Proposition 4 says that $f(b_1, ..., b_r; \lambda_1, ..., \lambda_r)$ (resp. $g(b_1, ..., b_r; \lambda_1, ..., \lambda_r)$) is equal to the $r \times r$ minor obtained by picking out the b_ith rows and λ_jth columns from the $n \times n$ matrix $A_n = (q^{3i + 3j^2 - 6j + 1} [{i-1 \atop j-1}]_6)_{1 \leqslant i,j \leqslant n}$ (resp. $B_n = (q^{i + \binom{j}{2}} [{i-1 \atop j-1}])_{1 \leqslant i,j \leqslant n}$). Therefore we obtain

THEOREM 2. *For a matrix* X, *let* $M(X)$ *denote the sum of all minors* (*including the void minor equal to* 1) *of* X. *Let* A_n *and* B_n *as above*:

(a) $P(S_3; n) = M(A_n)$.

(b) $Q(S_3; n) = M(B_n)$.

Remark. The equality (b) of Theorem 2 is noted in [St2, Conjecture 7] without proof.

3. Formulas for the Sum of Minors

Let $z(i,j)$ $(1\leqslant i\leqslant n,\ 1\leqslant j\leqslant m)$ be indeterminates and $Z=(z(i,j))$ be the $n\times m$ matrix with (i,j)-entry $z(i,j)$. The object of this section is to derive a formula for the sum $M(Z)$ of all minors of Z.

Definition. We shall write $d(a_1,\dots,a_r;\ b_1,\dots,b_r)=d_Z(a_1,\dots,a_r;\ b_1,\dots,b_r)$ for the determinant $\det(z(a_i,b_j))_{1\leqslant i,j\leqslant r}$. Define

$$d(a_1,\dots,a_r)=d_Z(a_1,\dots,a_r)=\sum d_Z(a_1,\dots,a_r;b_1,\dots,b_r)$$

summed over all increasing sequences $(1\leqslant)\ b_1<\cdots<b_r\ (\leqslant m)$.

If any two a_j are equal, then $d(a_1,\dots,a_r)=0$; otherwise $d(a_1,\dots,a_r)$ is equal to the sum of all $r\times r$ minors of the $r\times m$ matrix $(z(a_i,j))$.

For a $2k\times 2k$ alternating matrix $X=(x_{ij})_{1\leqslant i,j\leqslant 2k}$ $(x_{ij}+x_{ji}=0)$, there is a unique polynomial $\mathrm{Pf}_{2k}(X)$ in x_{ij} $(i<j)$, called Pfaffian, such that

(Pf1) $\det(X)=\mathrm{Pf}_{2k}(X)^2$

(Pf2) $\mathrm{Pf}_{2k}(J_k)=1$,

where J_k is the $2k\times 2k$ matrix,

$$J_k=\begin{pmatrix} 0 & 1 & & & & & 0 \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ 0 & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

If we denote by X_{ij} the $2(k-1)\times 2(k-1)$ alternating matrix obtained by deleting the ith row, the jth row, the ith column and the jth column of X, it is known that

$$\mathrm{Pf}_{2k}(X)=\sum_{i=2}^{2k}(-1)^i x_{1i}\mathrm{Pf}_{2(k-1)}(X_{1i}).$$

THEOREM 3. (a) *If r is odd, then*

$$d(a_1, ..., a_r) = \mathrm{Pf}_{r+1}\begin{pmatrix} 0 & d(a_1) & d(a_2) & d(a_3) & \cdots & d(a_r) \\ -d(a_1) & 0 & d(a_1, a_2) & d(a_1, a_3) & \cdots & d(a_1, a_r) \\ -d(a_2) & -d(a_1, a_2) & 0 & d(a_2, a_3) & \cdots & d(a_2, a_r) \\ -d(a_3) & -d(a_1, a_3) & -d(a_2, a_3) & 0 & \cdots & d(a_3, a_r) \\ & & \cdots & & & \\ -d(a_r) & -d(a_1, a_r) & -d(a_2, a_r) & -d(a_3, a_r) & \cdots & 0 \end{pmatrix}.$$

(b) If r is even, then

$$d(a_1, ..., a_r) = \mathrm{Pf}_r\begin{pmatrix} 0 & d(a_1, a_2) & d(a_1, a_3) & \cdots & d(a_1, a_r) \\ -d(a_1, a_2) & 0 & d(a_2, a_3) & \cdots & d(a_2, a_r) \\ -d(a_1, a_3) & -d(a_2, a_3) & 0 & \cdots & d(a_3, a_r) \\ & & \cdots & & \\ -d(a_1, a_r) & -d(a_2, a_r) & -d(a_3, a_r) & \cdots & 0 \end{pmatrix}.$$

Proof. First we note that part (a) follows from part (b) by applying (b) to the $(r+1)\times(m+1)$ matrix

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & z(a_i, j) & \\ 0 & & & \end{array}\right).$$

To prove part (b), we may assume $a_i = i$ for $i = 1, ..., r$. It is clear that $d(1, 2, ..., r)$ is multilinear and alternating with respect to row vectors $z_i = (z(i, 1), ..., z(i, m))$ of Z $(1 \leqslant i \leqslant r)$. Since $\mathrm{Pf}_r(x_{ij})$ is multilinear in $(x_{i1}, ..., x_{ir})$ and $\mathrm{Pf}_r(x_{ij}) = 0$ if any two rows of $X = (x_{ij})$ are equal, $\mathrm{Pf}_r(d(i, j))$ is also multilinear and alternating in z_i's. Therefore it is enough to check $d(1, 2, ..., r) = \mathrm{Pf}_r(d(i, j))$ for the case $z(i, j) = \delta_{i, i_j}$, where $\delta_{k,l}$ is the Kronecker delta and $i_1 < i_2 < \cdots < i_r$. But in this case, we see that

$$d(1, ..., r) = 1$$

and that

$$\mathrm{Pf}_r(d(i, j)) = 1$$

by using the expansion formula of Pfaffian and $d(i, j) = 1$ $(i < j)$.

COROLLARY. (a) *If r is odd,*

$$d(a_1, ..., a_r) = \sum_{i=1}^{r} (-1)^{i-1} d(a_i)\, d(a_1, ..., \widehat{a_i}, ..., a_r). \tag{5}$$

(b) *If r is even,*

$$d(a_1, ..., a_r) = \sum_{i=2}^{r} (-1)^{i} d(a_1, a_i)\, d(a_2, ..., \widehat{a_i}, ..., a_r). \tag{6}$$

(The symbol $\widehat{\ }$ means omitting a_i.)

Now we consider the sum $M(Z)$ of all minors (including the void minor) of the $n \times m$ matrix Z. Since $d(a_1, ..., a_r)$ is equal to the sum of $r \times r$ minors of the $r \times m$ matrix $(z(a_i, j))_{1 \leqslant i \leqslant r,\, 1 \leqslant j \leqslant m}$,

$$M(Z) = 1 + \sum_{1 \leqslant a_1 < a_2 < \cdots < a_r \leqslant n} d(a_1, a_2, ..., a_r).$$

The main theorem of this section is the following.

THEOREM 4. *For $Z = (z(i, j))_{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant m}$, we define an $(n+1) \times (n+1)$ alternating matrix $V(Z)$ and an $(n+2) \times (n+2)$ alternating matrix $\tilde{V}(Z)$ as*

$$V(Z) =$$

$$\begin{pmatrix} 0 & d(1)+1 & d(2)-1 & d(3)+1 & \cdots & d(n)+(-1)^{n-1} \\ -d(1)-1 & 0 & d(1,2)+1 & d(1,3)-1 & \cdots & d(1,n)+(-1)^{n} \\ -d(2)+1 & -d(1,2)-1 & 0 & d(2,3)+1 & \cdots & d(2,n)+(-1)^{n+1} \\ -d(3)-1 & -d(1,3)+1 & -d(2,3)-1 & 0 & \cdots & d(3,n)+(-1)^{n+2} \\ & & \cdots & & & \\ -d(n)-(-1)^{n-1} & -d(1,n)-(-1)^{n} & -d(2,n)-(-1)^{n+1} & -d(3,n)-(-1)^{n+2} & \cdots & 0 \end{pmatrix}$$

$$\tilde{V}(Z) = \left(\begin{array}{cccc|c} & & & & (-1)^{n} \\ & & & & (-1)^{n+1} \\ & V(Z) & & & (-1)^{n+2} \\ & & & & \vdots \\ & & & & (-1)^{2n} \\ \hline -(-1)^{n} & -(-1)^{n+1} & -(-1)^{n+2} \;\cdots & -(-1)^{2n} & 0 \end{array}\right).$$

Then

(a) *If n is odd,* $\quad M(Z) = \mathrm{Pf}_{n+1}(V(Z))$.

(b) *If n is even,* $\quad M(Z) = \mathrm{Pf}_{n+2}(\tilde{V}(Z))$.

In order to prove this theorem, we have to prepare some notations and lemmas. For a subset $S = \{a_1, ..., a_r\}$ $(a_1 < \cdots < a_r)$ of $[n] = \{1, 2, ..., n\}$, we shall write $\bar{d}(S)$ for $d(a_1, ..., a_r)$. If $S = \varnothing$, we put $\bar{d}(\varnothing) = 1$.

Fix a non-empty subset $S=\{a_1, ..., a_r\}$ of $[n]$ $(a_1 < \cdots < a_r)$. Let W_S be the $r\times r$ alternating matrix whose (i,j)-entry is $d(a_i, a_j)+(-1)^{j-i+1}$ $(i<j)$. And let $\tilde{W}_S$ be the following $(r+1)\times(r+1)$ alternating matrix:

$$\tilde{W}_S=\left(\begin{array}{c|cccc} 0 & d(a_1)+1 & d(a_2)-1 & \cdots & d(a_r)+(-1)^{r-1} \\ \hline -d(a_1)-1 & & & & \\ -d(a_2)+1 & & & & \\ \vdots & & & W_S & \\ -d(a_r)-(-1)^{r-1} & & & & \end{array}\right).$$

Let $I=\{i_1, ..., i_k\}$ $(i_1 < \cdots < i_k)$ be a subset of $[r]=\{1, ..., r\}$. We put $s(I)=i_1+\cdots+i_k$. For a subset $J=\{i_{j(1)}, ..., i_{j(l)}\}$ $(j(1)<\cdots<j(l))$ of I, we put $s_I(J)=j(1)+\cdots+j(l)$. If I or J is empty, we define $s(\varnothing)=s_{\varnothing}(\varnothing)=0$ and $s_I(\varnothing)=0$. Moreover, we define, for $J\subset I$,

$$\varepsilon_I(J)=(-1)^{s(I-J)+s_I(I-J)}.$$

The following two lemmas are the key to the proof of Theorem 4.

LEMMA 1. *Let $I=\{i_1, ..., i_k\}$ $(i_1<\cdots<i_k)$ be the subset of $[r]$, and let $W_S(I)$ denote the $k\times k$ alternating matrix obtained by picking out the i_a-rows and the i_b-columns from W_S. If k is even, then*

$$\mathrm{Pf}_k(W_S(I))=\sum_{\substack{J\subset I\\ \sharp J \text{ is even}}} \varepsilon_I(J)\,\tilde{d}(S_J), \tag{7}$$

where $S_J=\{a_{j(1)}, ..., a_{j(l)}\}$ $(J=\{i_{j(1)}, ..., i_{j(l)}\}$ $(j(1)<\cdots<j(l)))$ and $S_{\varnothing}=\varnothing$.

LEMMA 2. *If $r=\sharp S$ is odd, then*

$$\mathrm{Pf}_{r+1}(\tilde{W}_S)=\sum_{T\subset S}\tilde{d}(T).$$

Proof of Lemma 1. We proceed by induction on $k=\sharp I$. When $k=2$, $\mathrm{Pf}_2(W_S(I))=d(a_{i_1}, a_{i_2})+(-1)^{i_1+i_2}$. But $\varepsilon_I(I)=1$ and $\varepsilon_I(\varnothing)=(-1)^{i_1+i_2}$. Hence (7) holds for the case $k=2$.

When $k>2$, expanding $\mathrm{Pf}_k(W_S(I))$ along the first row, we have

$$\begin{aligned}\mathrm{Pf}_k(W_S(I)) &= \sum_{p=2}^{k}(-1)^p\,(d(a_{i_1}, a_{i_p})+(-1)^{i_1+i_p+1}) \\ &\quad\times \mathrm{Pf}_{k-2}(W_S(I-\{i_1, i_p\})) \\ &= \sum_{p=2}^{k}(-1)^p\,(d(a_{i_1}, a_{i_p})+(-1)^{i_1+i_p+1}) \\ &\quad\times \sum_{\substack{K\subset I-\{i_1, i_p\}\\ \sharp K\text{ is even}}} \varepsilon_{I-\{i_1, i_p\}}(K)\,\tilde{d}(S_K) \end{aligned} \tag{8}$$

by induction hypothesis. We shall compute the coefficient $c(J)$ of $\tilde{d}(S_J)$ ($J \subset I$ and $\sharp J$ is even) in the right-hand side of (8).

In the case $i_1 \notin J$, if we put $I - J = \{i_{h(1)}, ..., i_{h(l')}\}$ $(h(1) < \cdots < h(l'))$, we have

$$c(J) = \sum_{t=2}^{l'} (-1)^{h(t)} \cdot (-1)^{i_1 + i_{h(t)} + 1} \varepsilon_{I - \{i_1, i_{h(t)}\}}(J).$$

But it follows from the definition that

$$s((I - \{i_1, i_{h(t)}\}) - J) = s(I - J) - i_1 - i_{h(t)},$$

$$s_{I - \{i_1, i_{h(t)}\}}((I - \{i_1, i_{h(t)}\}) = s_I(I - J) - h(t) + 2l' + t + 1.$$

Hence

$$c(J) = \sum_{t=2}^{l'} (-1)^{s(I-J) + s_I(I-J) + t} = \varepsilon_I(J) \cdot \sum_{t=2}^{l'} (-1)^t = \varepsilon_I(J)$$

because $l' = \sharp(I - J)$ is even.

In the case $i_1 \in J$, let $J = \{i_{j(1)}, ..., i_{j(l)}\}$ $(j(1) < \cdots < j(l))$. The coefficient $c(J; t)$ of $d(a_{i_1}, a_{i_{j(t)}})\, \tilde{d}(S_{J - \{i_1, i_{j(t)}\}})$ in the right-hand side of (8) is equal to

$$(-1)^{j(t)} \varepsilon_{I - \{i_1, i_{j(t)}\}} (J - \{i_1, i_{j(t)}\}).$$

But

$$\begin{aligned} s((I - \{i_1, i_{j(t)}\}) - (J - \{i_1, i_{j(t)}\})) &= s(I - J) \\ s_{I - \{i_1, i_{j(t)}\}}((I - \{i_1, i_{j(t)}\}) - (J - \{i_1, i_{j(t)}\})) & \\ = s_I(I - J) + j(t) - 2k + 2l - t. & \end{aligned}$$

Hence we have

$$c(J; t) = (-1)^t \varepsilon_I(J).$$

It follows from (6) (Corollary (b), Theorem 3) that

$$c(J) = \varepsilon_I(J)$$

and that the right-hand side of (8) is a linear combination of the $\tilde{d}(S_J)$'s ($J \subset I$ and $\sharp J$ is even). Therefore (7) is valid for any even k.

Proof of Lemma 2. Expanding $\mathrm{Pf}_{r+1}(\tilde{W}_S)$ along the first row, we see that

$$\begin{aligned}\mathrm{Pf}_{r+1}(W_S) &= \sum_{i=1}^{r} (-1)^{i+1} (d(a_i)+(-1)^{i+1})\, \mathrm{Pf}_{r-1}(W_S([r]-\{i\})) \\ &= \sum_{i=1}^{r} (-1)^{i+1} (d(a_i)+(-1)^{i+1}) \\ &\quad \times \sum_{\substack{K\subset [r]-\{i\} \\ \sharp K \text{ is even}}} \varepsilon_{[r]-\{i\}}(K)\, \tilde{d}(S_K) \qquad (9)\end{aligned}$$

by applying Lemma 1 to the case $I=[r]-\{i\}$. We shall show that the coefficient $c'(T)$ of $\tilde{d}(T)$ $(T\subset S)$ in (9) is equal to 1. Let I be the subset of $[r]$ such that $S_I=T$.

When $\sharp T=\sharp I$ is even,

$$c'(T)=\sum_{t=1}^{k'} (-1)^{j_t+1}\cdot(-1)^{j_t+1}\, \varepsilon_{[r]-\{j_t\}}(I),$$

where $[r]-I=\{j_1, ..., j_{k'}\}$ $(j_1<\cdots<j_{k'})$. Since

$$s(([r]-\{j_t\})-I)=s([r]-I)-j_t$$
$$s_{[r]-\{j_t\}}(([r]-\{j_t\})-I)=s([r]-I)-j_t-(k'-t),$$

we obtain

$$c'(T)=\sum_{t=1}^{k'} (-1)^{k'-t}=1,$$

because $k'=r-\sharp T$ is odd.

When $\sharp T=\sharp I$ is odd, let $I=\{i_1, ..., i_k\}$ $(i_1<\cdots<i_k)$. The coefficient $c'(T; t)$ of $d(a_{i_t})\,\tilde{d}(S_{I-\{i_t\}})$ in the right-hand side of (9) is equal to

$$(-1)^{i_t+1}\, \varepsilon_{[r]-\{i_t\}}(I-\{i_t\}).$$

But

$$s(([r]-\{i_t\})-(I-\{i_t\}))=s([r]-I)$$
$$s_{[r]-\{i_t\}}(([r]-\{i_t\})-(I-\{i_t\})=s([r]-\{i_t\})+i_t-r+k-t.$$

Since r and k are even, we have

$$c'(T; t)=(-1)^{r+k+t+1}=(-1)^{t+1}.$$

It follows from (5) (Corollary (a), Theorem 3) that $c'(T)=1$ and that the right-hand side of (9) is a linear combination of the $\tilde{d}(T)$'s $(T\subset S)$. Therefore the proof of Lemma 2 is completed.

Proof of Theorem 4. Note that

$$M(Z)=\sum_{T\subset[n]}\tilde{d}(T).$$

(a) When n is odd, by applying Lemma 2 to $S=[n]$, we have

$$M(Z)=\mathrm{Pf}_{n+1}(W_{[n]})=\mathrm{Pf}_{n+1}(V(Z)),$$

because $W_{[n]}=V(Z)$.

(b) When n is even, by applying part (a) to the $(n+1)\times m$ matrix

$$Z'=\begin{pmatrix} z(1,1) & z(1,2) & \cdots & z(1,m)\\ z(2,1) & z(2,2) & \cdots & z(2,m)\\ & \cdots & & \\ z(n,1) & z(n,2) & \cdots & z(n,m)\\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

we see that $d_{Z'}(n+1)=d_{Z'}(i,n+1)=0$ and

$$M(Z)=M(Z')=\mathrm{Pf}_{n+2}(\tilde{V}(Z)).$$

This completes the proof of Theorem 4.

4. Application of the Minor-Sum Formula to the Generating Functions for Plane Partitions

In Theorem 2 (Section 2), we have shown that

$$P(S_3;n)=M(A_n)$$

$$Q(S_3;n)=M(B_n),$$

where $M(X)$ denotes the sum of all minors (including the void minor equal to 1) of X and

$$A_n=\left(q^{3i+3j^2-6j+1}\begin{bmatrix} i-1\\ j-1\end{bmatrix}_6\right)_{1\leqslant i,j\leqslant n}$$

$$B_n=\left(q^{i+\binom{j}{2}}\begin{bmatrix} i-1\\ j-1\end{bmatrix}\right)_{1\leqslant i,j\leqslant n}.$$

Therefore, applying the minor-sum formula of Theorem 4 (Section 3), we have

$$P(S_3;n)=\begin{cases}\mathrm{Pf}_{n+1}(V(A_n)) & (\text{if } n \text{ is odd})\\ \mathrm{Pf}_{n+2}(\tilde{V}(A_n)) & (\text{if } n \text{ is even})\end{cases}$$

$$Q(S_3;n)=\begin{cases}\mathrm{Pf}_{n+1}(V(B_n)) & (\text{if } n \text{ is odd})\\ \mathrm{Pf}_{n+2}(\tilde{V}(B_n)) & (\text{if } n \text{ is even}),\end{cases}$$

where $V(Z)$ and $\tilde{V}(Z)$ are the alternating matrices defined in Theorem 4. In order to compute the above Pfaffians, we need information about

$$d_{A_n}(i)=\sum_{j\geq 1} q^{3i+3j^2-6j+1}\begin{bmatrix} i-1\\ j-1\end{bmatrix}_6,$$

$$d_{A_n}(i_1,i_2)$$

$$=\sum_{1\leq j_1<j_2}\det\begin{pmatrix} q^{3i_1+3j_1^2-6j_1+1}\begin{bmatrix} i_1-1\\ j_1-1\end{bmatrix}_6 & q^{3i_1+3j_2^2-6j_2+1}\begin{bmatrix} i_1-1\\ j_2-1\end{bmatrix}_6\\ q^{3i_2+3j_1^2-6j_1+1}\begin{bmatrix} i_2-1\\ j_1-1\end{bmatrix}_6 & q^{3i_2+3j_2^2-6j_2+1}\begin{bmatrix} i_2-1\\ j_2-1\end{bmatrix}_6\end{pmatrix},$$

$$d_{B_n}(i)=\sum_{j\geq 1} q^{i+\binom{j}{2}}\begin{bmatrix} i-1\\ j-1\end{bmatrix},$$

$$d_{B_n}(i_1,i_2)$$

$$=\sum_{1\leq j_1\leq j_2}\det\begin{pmatrix} q^{i_1+\binom{j_1}{2}}\begin{bmatrix} i_1-1\\ j_1-1\end{bmatrix} & q^{i_1+\binom{j_2}{2}}\begin{bmatrix} i_1-1\\ j_2-1\end{bmatrix}\\ q^{i_2+\binom{j_1}{2}}\begin{bmatrix} i_2-1\\ j_1-1\end{bmatrix} & q^{i_2+\binom{j_2}{2}}\begin{bmatrix} i_2-1\\ j_2-1\end{bmatrix}\end{pmatrix}.$$

Proposition 5. (a) $d_{A_n}(i+1)=q^3(1+q^{6i-3})\,d_{A_n}(i)$, $d_{A_n}(1)=q$.

(b) $d_{B_n}(i+1)=q(1+q^i)\,d_{B_n}(i)$, $d_{B_n}(1)=q$.

(c) $d_{A_n}(1,i)=qd_{A_n}(i)-q^{3i-1}$

(d) $d_{B_n}(1,i)=qd_{B_n}(i)-q^{i+1}$

(e) $d_{A_n}(i_1,i_2+1)=q^3(1+q^{6i_2-3})\,d_{A_n}(i_1,i_2)$

$$+q^{3i_1+9i_2-4}\begin{bmatrix} i_1+i_2-2\\ i_1-1\end{bmatrix}_6+q^{3i_1+9i_2-1}\begin{bmatrix} i_1+i_2-2\\ i_1-2\end{bmatrix}_6$$

$d_{A_n}(i_1+1,i_2)=q^3(1+q^{6i_1-3})\,d_{A_n}(i_1,i_2)$

$$-q^{9i_1+3i_2-4}\begin{bmatrix} i_1+i_2-2\\ i_2-1\end{bmatrix}_6-q^{9i_1+3i_2-1}\begin{bmatrix} i_1+i_2-2\\ i_2-2\end{bmatrix}_6$$

(f) $d_{B_n}(i_1,i_2+1)=q(1+q^{i_2})\,d_{B_n}(i_1,i_2)+q^{i_1+2i_2+1}\begin{bmatrix} i_1+i_2-1\\ i_1-1\end{bmatrix}$

$d_{B_n}(i_1+1,i_2)=q(1+q^{i_1})\,d_{B_n}(i_1,i_2)-q^{2i_1+i_2+1}\begin{bmatrix} i_1+i_2-1\\ i_2-1\end{bmatrix}$

Proof. We shall prove only (b), (d), and (f). The other parts (a), (c), and (e) can be similarly proved.

(b) If $i \geqslant 1$, we have

$$\sum_{k=1}^{i} d_{B_n}(k) = \sum_{k=1}^{i} \sum_{j=1}^{k} q^{k+\binom{j}{2}} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}$$
$$= \sum_{j=1}^{i} \sum_{k=j}^{i} q^{\binom{j+1}{2}+k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}$$
$$= \sum_{j=1}^{i} q^{\binom{j+1}{2}} \begin{bmatrix} i \\ j \end{bmatrix}$$

because $\sum_{k=j}^{i} q^{k-j} \left[\begin{smallmatrix} k-1 \\ j-1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]$. Hence

$$\sum_{k=1}^{i} d_{B_n}(k) = -1 + q^{-(i+1)} d_{B_n}(i+1).$$

Since this holds for $i=0$,

$$d_{B_n}(i) = \sum_{k=1}^{i} d_{B_n}(k) - \sum_{k=1}^{i-1} d_{B_n}(k)$$
$$= -1 + q^{-(i+1)} d_{B_n}(i+1) - (-1 + q^{-i} d_{B_n}(i)),$$

so that we obtain

$$d_{B_n}(i+1) = q(1+q^i)\, d_{B_n}(i).$$

(d) It follows from the definition that

$$d_{B_n}(1, i) = \sum_{j_2 > 1} q \cdot q^{i+\binom{j_2}{2}} \begin{bmatrix} i-1 \\ j_2-1 \end{bmatrix} = q \cdot d_{B_n}(i) - q^{i+1}.$$

(f) If $m \geqslant 1$, we see that

$$\sum_{j=1}^{m} d_{B_n}(i, j)$$
$$= \sum_{j=1}^{m} \sum_{1 \leqslant k < l} q^{i+j+\binom{k}{2}+\binom{l}{2}}$$
$$\times \left(\begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \begin{bmatrix} j-1 \\ l-1 \end{bmatrix} - \begin{bmatrix} i-1 \\ l-1 \end{bmatrix} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix} \right)$$
$$= \sum_{1 \leqslant k < l} q^{i+\binom{k}{2}+\binom{l+1}{2}} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \sum_{j=1}^{m} q^{j-l} \begin{bmatrix} j-1 \\ l-1 \end{bmatrix}$$
$$- \sum_{1 \leqslant k < l} q^{i+\binom{k+1}{2}+\binom{l}{2}} \begin{bmatrix} i-1 \\ l-1 \end{bmatrix} \sum_{j=k}^{m} q^{j-k} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}$$

$$
\begin{aligned}
&= \sum_{1 \leqslant k < l} q^{i + \binom{k}{2} + \binom{l+1}{2}} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} \\
&\quad - \sum_{1 \leqslant k < l} q^{i + \binom{k+1}{2} + \binom{l}{2}} \begin{bmatrix} i-1 \\ l-1 \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \\
&= \sum_{1 \leqslant k < l'} q^{i + \binom{k}{2} + \binom{l'}{2}} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \begin{bmatrix} m \\ l'-1 \end{bmatrix} \\
&\quad - \sum_{k \geqslant 1} q^{i + \binom{k}{2} + \binom{k+1}{2}} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \\
&\quad - \Bigg(\sum_{1 < k' < l} q^{i + \binom{k'}{2} + \binom{l}{2}} \begin{bmatrix} i-1 \\ l-1 \end{bmatrix} \begin{bmatrix} m \\ k'-1 \end{bmatrix} \\
&\quad + \sum_{k' \geqslant 1} q^{i + \binom{k'}{2} + \binom{k'}{2}} \begin{bmatrix} i-1 \\ k'-1 \end{bmatrix} \begin{bmatrix} m \\ k'-1 \end{bmatrix} \\
&\quad - \sum_{l \geqslant 1} q^{i + \binom{l}{2}} \begin{bmatrix} i-1 \\ l-1 \end{bmatrix} \Bigg) \\
&= \sum_{1 \leqslant k < l} q^{i + \binom{k}{2} + \binom{l}{2}} \left(\begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \begin{bmatrix} m \\ l-1 \end{bmatrix} - \begin{bmatrix} i-1 \\ l-1 \end{bmatrix} \begin{bmatrix} m \\ k-1 \end{bmatrix} \right) \\
&\quad - \sum_{k \geqslant 1} q^{i + k(k-1)} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \begin{bmatrix} m+1 \\ k \end{bmatrix} + \sum_{l \geqslant 1} q^{i + \binom{l}{2}} \begin{bmatrix} i-1 \\ l-1 \end{bmatrix}.
\end{aligned}
$$

But the first term is $q^{-(m+1)} d_{B_n}(i, m+1)$ and the third term is $d_{B_n}(i)$. Moreover, it follows from $\sum_{k=0}^{h} q^{(n-k)(h-k)} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \left[\begin{smallmatrix} m \\ h-k \end{smallmatrix}\right] = \left[\begin{smallmatrix} m+n \\ h \end{smallmatrix}\right]$([An1, p. 37, (3.3.10)]) that the second term is equal to $q^i \left[\begin{smallmatrix} i+m \\ i \end{smallmatrix}\right]$. Therefore we obtain

$$
\sum_{j=1}^{m} d_{B_n}(i, j) = q^{-(m+1)} d_{B_n}(i, m+1) + d_{B_n}(i) - q^i \begin{bmatrix} i+m \\ i \end{bmatrix}.
$$

And (c) implies that this holds for $m = 0$. Hence we have

$$
\begin{aligned}
d_{B_n}(i_1, i_2) &= \sum_{j=1}^{i_2} d_{B_n}(i_1, j) - \sum_{j=1}^{i_2-1} d_{B_n}(i_1, j) \\
&= q^{-(i_2+1)} d_{B_n}(i_1, i_2+1) - q^{-i_2} d_{B_n}(i_1, i_2) \\
&\quad - q^{i_1} \left(\begin{bmatrix} i_1 + i_2 \\ i_1 \end{bmatrix} - \begin{bmatrix} i_1 + i_2 - 1 \\ i_1 \end{bmatrix} \right)
\end{aligned}
$$

so that

$$
d_{B_n}(i_1, i_2+1) = q(1 + q^{i_2}) \, d_{B_n}(i_1, i_2) + q^{i_1 + 2i_2 + 1} \begin{bmatrix} i_1 + i_2 - 1 \\ i_1 - 1 \end{bmatrix}
$$

by using

$$\begin{bmatrix} i_1+i_2 \\ i_1 \end{bmatrix} - \begin{bmatrix} i_1+i_2-1 \\ i_1 \end{bmatrix} = q^{i_2} \begin{bmatrix} i_1+i_2-1 \\ i_1-1 \end{bmatrix}.$$

Now we can use Proposition 5 to simplify the Pfaffians $\mathrm{Pf}_{n+1}(V(A_n))$, $\mathrm{Pf}_{n+2}(\tilde{V}(A_n))$, $\mathrm{Pf}_{n+1}(V(B_n))$, and $\mathrm{Pf}_{n+2}(\tilde{V}(B_n))$. Here we deal with only $\mathrm{Pf}_{n+1}(V(B_n))$ and $\mathrm{Pf}_{n+2}(\tilde{V}(B_n))$. The similar method can be applied to $\mathrm{Pf}_{n+1}(V(A_n))$ and $\mathrm{Pf}_{n+2}(\tilde{V}(A_n))$. In the $(n+1)\times(n+1)$ matrix $V(B_n)$ (resp. the $(n+2)\times(n+2)$ matrix $\tilde{V}(B_n)$), we start from the nth column, subtract from each column (say, the ith column) the previous column multiplied by $q(1+q^{i-1})$ $(i=n, n-1, ..., 2)$ and finally subtract the 1st column multiplied by q from the 2nd column. Then, if we denote the resulting matrix $V'_n=(v'_{ij})_{1\leqslant i,j\leqslant n+1}$ (resp. $\tilde{V}'_n=(\tilde{v}'_{ij})_{1\leqslant i,j\leqslant n+2}$), it follows from Proposition 5 (b), (d), and (f) that

$$v'_{ij}=\begin{cases} 0 & (i=1, j=1) \\ 1+q & (i=1, j=2) \\ (-1)^j(1+q+q^{j-1}) & (i=1, j\geqslant 3) \\ -d_{B_n}(i-1)-(-1)^i & (i\geqslant 2, j=1) \\ q+q^2 & (i=2, j=2) \\ (-1)^i(1+q^i)+q & (i\geqslant 3, j=2) \\ \lambda_{ij}+q^{i+2j-4}\begin{bmatrix} i+j-4 \\ i-2 \end{bmatrix} & (i\geqslant 2, j\geqslant 3), \end{cases}$$

where λ_{ij} is given by

$$\lambda_{ij}=\begin{cases} (-1)^{i-j+1}(1+q+q^{j-1}), & (j>i+1) \\ 1 & (j=i+1) \\ q+q^{j-1} & (j=i) \\ (-1)^{i-j}(1+q+q^{j-1}) & (j<i) \end{cases}$$

and that

$$\tilde{V}'_n=\left(\begin{array}{cccc|c} & & & & (-1)^n \\ & & & & (-1)^{n+1} \\ & & V'_n & & (-1)^{n+2} \\ & & & & \vdots \\ & & & & (-1)^{2n} \\ \hline (-1)^{n+1} & (-1)^{n+2}(1+q) & (-1)^{n+3}(1+q+q^2) & \cdots (-1)(1+q+q^n) & (-1)^{2n+1} \end{array}\right).$$

Let V_n^* be the upper-right $(n+1)\times(n+1)$ submatrix of $\tilde{V}_n'$. Since $\tilde{v}_{n+2,j}' = (-1)^n v_{1,j}'$ $(2 \leqslant j \leqslant n+1)$,

$$\begin{aligned}\det \tilde{V}_n' &= (-1)^{n+2+1}\cdot(-1)^{n+1}\det V_n^* + (-1)^{n+2+n+2}\cdot(-1)^{2n+1}\det V_n' \\ &= \det V_n^* - \det V_n'\end{aligned}$$

by adding the first row multuplied by $(-1)^{n+1}$ to the $(n+2)$nd row and then expanding along the $(n+2)$nd row. Hence we see that

$$\det \tilde{V}(B_n) = \det V_n^* - \det V(B_n).$$

But, if n is odd, $\tilde{V}(B_n)$ is an alternating matrix of odd degree, so that

$$\det V(B_n) = \det V_n^*.$$

If n is even, $V(B_n)$ is an alternating matrix of odd degree, so that

$$\det \tilde{V}(B_n) = \det V_n^*.$$

Therefore we obtain

$$Q(S_3; n)^2 = \det V_n^*$$

whether n is odd or even. By starting from the nth row and adding each row to the one below it in V_n^*, we see that

$$\det V_n^* = \det(U_n^{(1)} + U_n^{(2)}),$$

where

$$U_n^{(1)} = \begin{pmatrix} 1+q & -q-q^2 & & & & & 0 \\ -1 & 1+q+q^2 & -q-q^3 & & & & \\ & -1 & 1+q+q^3 & -q-q^4 & & & \\ & & -1 & 1+q+q^4 & & & \\ & 0 & & & \ddots & & -q-q^n \\ & & & & & -1 & 1+q+q^n \end{pmatrix}$$

and, if we set $U_n^{(2)} = (u_{ij})_{1 \leqslant i,j \leqslant n}$,

$$u_{ij} = \begin{cases} q^i(1+q) & (j=1) \\ q^{i+2j-2}\left(\begin{bmatrix} i+j-3 \\ i-2 \end{bmatrix} + q\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}\right) & (j \geqslant 2). \end{cases}$$

Now, by starting from the first column and adding each column multiplied by q to the next column, we obtain the following formula.

THEOREM 5. *$Q(S_3;n)^2=\det(T_n^{(1)}+T_n^{(2)})$, where $T_n^{(1)}$ and $T_n^{(2)}$ are $n\times n$ matrices*

$$T_n^{(1)}=\begin{pmatrix} 1+q & & & & & 0 \\ -1 & 1+q^2 & & & & \\ & -1 & 1+q^3 & & & \\ & & -1 & 1+q^4 & & \\ & 0 & & \ddots & \ddots & \\ & & & & -1 & 1+q^n \end{pmatrix}$$

$$T_n^{(2)}=\left(q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}+q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}\right)\right)_{1\leqslant i,j\leqslant n}.$$

This theorem reduces the Andrews–Robbins conjecture (see the Introduction) to the equality

$$\det(T_n^{(1)}+T_n^{(2)})=\prod_{1\leqslant i\leqslant j\leqslant k\leqslant n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2.$$

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